# THERMOELASTIC STRESSES IN A LONG HOLLOW CYLINDER UNDER LOCAL HEATING OF THE SIDE SURFACE ACCORDING TO A NEWTON LAW

P. Z. Livshits

UDC 539.30:536.248

The three-dimensional thermoelasticity problem for a hollow isotropic cylinder is considered. The solution is obtained in a form effective for computations for a given discontinuous temperature field.

Let us consider the case of heating of a cylinder by the surrounding medium ($T_a > T$) when the heat exchange conditions on the outer ($r = a$) and inner ($r = b$) surfaces have the form

$$
\begin{aligned}
r = a:&\quad \frac{\partial T}{\partial r} + h_a(T - T_a) = 0, \quad T_a = \begin{cases} T_n \cos n\varphi, & |z| < c, \\ 0, & |z| > c, \end{cases} \\
r = b:&\quad \frac{\partial T}{\partial r} - h_b(T - T_b) = 0, \quad T_b = 0, \quad -\infty < z < \infty.
\end{aligned}
\tag{1}
$$

In the dimensionless coordinates $\rho$ and $\xi$ the solution of the stationary heat conduction problem satisfying the boundary conditions (1) can be written as a Fourier integral

$$
T(\rho, \varphi, \xi) = T_n \frac{2}{\pi} \cos n\varphi \int_0^\infty \left\{\left[1 - \frac{\beta}{\mathrm{Bi}_b} \frac{\partial}{\partial \beta}\right] L(\gamma\rho, \beta)\right\} \frac{\sin \gamma l \cos \gamma \xi}{\gamma \Lambda(\gamma)} d\gamma,
\tag{2}
$$

$$
\Lambda(\gamma) \equiv \Lambda(\lambda, \gamma) = \left[1 + \frac{\alpha}{\mathrm{Bi}_a} \frac{\partial}{\partial \alpha}\right]\left[1 - \frac{\beta}{\mathrm{Bi}_b} \frac{\partial}{\partial \beta}\right] L(\alpha, \beta) \equiv [+][-]L,
\tag{3}
$$

and the stresses $\sigma^{(T)}$ caused by the temperature field (2) can be represented thus [1]:

$$
\begin{aligned}
&\sigma_\rho^{(T)} = \sigma_\varphi^{(T)} = -\sigma_z^{(T)} = -\alpha_T E T, \\
&\tau_{\rho z}^{(T)} = \alpha_T E \frac{\partial}{\partial \rho} \int T d\xi, \quad \tau_{\varphi z}^{(T)} = \frac{\alpha_T E}{\rho} \frac{\partial}{\partial \varphi} \int T d\xi, \quad \tau_{\rho\varphi}^{(T)} = 0.
\end{aligned}
\tag{4}
$$

In solving the elasticity theory problem, we select the harmonic functions $\chi(\rho, \varphi, \xi)$ in terms of which the stresses $\sigma^{(Y)}$ are expressed in conformity with Hooke's law, in the form

$$
\begin{pmatrix} \chi_1, \chi_2 \\ \chi_3 \end{pmatrix} = \frac{2(1+\nu)}{\pi E} R^2 \begin{pmatrix} \cos \\ \sin \end{pmatrix} n\varphi \int_0^\infty \left[\begin{pmatrix} A_1, A_2 \\ A_3 \end{pmatrix} I(\gamma\rho) + \begin{pmatrix} A_4, A_5 \\ A_6 \end{pmatrix} K(\gamma\rho)\right] \frac{\sin \gamma l \cos \gamma \xi}{\gamma} d\gamma,
\tag{5}
$$

where the constants $A_1(\gamma), \ldots, A_6(\gamma)$ are to be determined from the conditions of no total ($\sigma^{(T)} + \sigma^{(Y)}$) stresses $\sigma_\rho$, $\tau_{\rho z}$, $\tau_{\rho\varphi}$ on the side surfaces ($\rho = 1 + \lambda$ and $\rho = 1 - \lambda$) of the cylinder. Compliance with these conditions according to (4) and (2), results at once, in a system of linear equations whose expanded matrix is ($\Theta \equiv \Theta_n = [\alpha_T E T_n]$)

$$
\begin{Vmatrix}
-V_I(\alpha), & I''(\alpha), & n\left[\dfrac{I(\alpha)}{\alpha}\right]', & \ldots, & \dfrac{\Theta}{\Lambda}[-]L \\
W_I(\alpha), & -I'(\alpha), & -n\dfrac{I(\alpha)}{2\alpha}, & \ldots, & -\dfrac{\Theta}{\Lambda}[-]L'_\alpha
\end{Vmatrix}
$$

Chemical-Pharmaceutical Institute, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 5, pp. 883-891, May, 1970. Original article submitted May 5, 1969.

$$\begin{Vmatrix} \frac{n}{2} I(\alpha), & -n\left[\frac{I(\alpha)}{\alpha}\right]', & \frac{I(\alpha)}{2} - I''(\alpha), & \ldots, & 0 \\ -V_I(\beta), & I''(\beta), & n\left[\frac{I(\beta)}{\beta}\right]', & \ldots, & \frac{\Theta}{\Lambda}\{[-]L(\gamma\rho, \beta)\}_\beta \\ W_I(\beta), & -I'(\beta), & -n\frac{I(\beta)}{2\beta}, & \ldots, & -\frac{\Theta}{\Lambda}\{[-]L'_{\gamma\rho}(\gamma\rho, \beta)\}_\beta \\ \frac{n}{2} I(\beta), & -n\left[\frac{I(\beta)}{\beta}\right]', & \frac{I(\beta)}{2} - I''(\beta), & \ldots, & 0 \end{Vmatrix} \tag{6}$$

The elements $a_{m,k}$ of the three columns not written down in (6) (k = 4, 5, 6) agree with the elements $a_{m,k-3}$ if the modified Bessel functions I in these latter are replaced by functions K of the same arguments.

Taking account of (4), (2), (5) and introducing the notation

$$F_N^{(T)}(\rho, \lambda, \gamma), \quad F_N^{(Y)}(\rho, \lambda, \gamma), \quad F_N(\rho, \lambda, \gamma) \qquad (N = 1, 2, \ldots, 6), \tag{7}$$

$$F_1^{(T)} = F_2^{(T)} = -F_3^{(T)} = \frac{\gamma\rho}{n} F_5^{(T)} = -\frac{[-]L(\gamma\rho, \beta)}{\gamma\Lambda(\gamma)}, \quad F_4^{(T)} = \frac{[-]L'_{\gamma\rho}(\gamma\rho, \beta)}{\gamma\Lambda(\gamma)}, \tag{8}$$

$$F_N^{(Y)} = \left[F_N^{(1)}[-]L - F_N^{(2)}[-]L'_\alpha + F_N^{(4)}\frac{1}{\mathrm{Bi}_b} - F_N^{(5)}\frac{1}{\beta}\right]\frac{1}{\Lambda(\gamma)}, \tag{9}$$

$$F_N = F_N^{(T)} + F_N^{(Y)},$$

we obtain a solution of the thermoelasticity problem in the form of the following integrals (N = 1 and N = 4):

$$\frac{\sigma_\rho}{\Theta\cos n\varphi} = \frac{2}{\pi}\int_0^\infty F_1(\rho, \lambda, \gamma)\sin\gamma l\cos\gamma\xi d\gamma,$$
$$\frac{\tau_{\rho z}}{\Theta\cos n\varphi} = \frac{2}{\pi}\int_0^\infty F_4(\rho, \lambda, \gamma)\sin\gamma l\sin\gamma\xi d\gamma. \tag{10}$$

The functions $F_N^{(m)}$ (m = 1, 2, 4, 5) in (8) have the following form for N = 1, say

$$F_1^{(m)} = \Bigg\{\Bigg(-A_{m1}V_I(\gamma\rho) + A_{m2}I''(\gamma\rho) + A_{m3}n\left[\frac{I(\gamma\rho)}{\gamma\rho}\right]'$$
$$-A_{m4}V_K(\gamma\rho) + A_{m5}K''(\gamma\rho) + A_{m6}n\left[\frac{K(\gamma\rho)}{\gamma\rho}\right]'\Bigg)(-\gamma^2)\Bigg\}\frac{1}{\gamma\Delta(\gamma)} \equiv \{\cdots\}\frac{1}{\gamma\Delta(\gamma)} \equiv \frac{\sigma_\rho^{(m)}(\rho, \lambda, \gamma)}{\Delta(\lambda, \gamma)}, \tag{11}$$

where $A_{m1}(\gamma), \ldots, A_{m6}(\gamma)$ are the algebraic cofactors of the elements of the m-th row of the system determinant (see [6]). The value of this determinant multiplied by $(-\gamma^2)$ is denoted by $\Delta(\gamma) \equiv \Delta_n(\lambda, \gamma)$ in (11).

Integrals such as (10) are evaluated on the basis of the Cauchy theorem by summing the residues (res) of the integrands over all poles $\gamma_s$ in the upper half-plane of $\gamma$:

$$\operatorname{res} F_N(\rho, \lambda, \gamma) = \lim_{\gamma\to\gamma_s}(\gamma - \gamma_s)F_N(\rho, \lambda, \gamma), \tag{12}$$

where $\gamma_s \equiv \gamma_{ns} = \pm i\varkappa_s$ are the roots of the equation $\Lambda_n(\lambda, \gamma) = 0$ (3); $\gamma_s \equiv \gamma_{ns} = \pm i\mu_s$, $\gamma_s \equiv \gamma_{ns} = \pm\eta_s \pm i\theta_s$ are the roots of the equation* $\Delta_n(\lambda, \gamma) = 0$ [see (6)].

The expressions (10) for the stresses can now be represented as:

$$\sigma_\rho = \sigma_\rho^0 + \Theta\Omega_1^T\cos n\varphi, \quad \sigma_z = \sigma_z^0 + \Theta\Omega_3^T\cos n\varphi, \quad \tau_{\varphi z} = \Theta\Omega_5^T\sin n\varphi,$$
$$\sigma_\varphi = \sigma_\varphi^0 + \Theta\Omega_2^T\cos n\varphi, \quad \tau_{\rho z} = \Theta\Omega_4^T\cos n\varphi, \quad \tau_{\rho\varphi} = \tau_{\rho\varphi}^0 + \Theta\Omega_6^T\sin n\varphi, \tag{13}$$

where the main components, being values of the residues at the pole $\gamma = 0$ (and corresponding to the solution which is obtained in the temperature distribution (2) as $\gamma \to 0$) equal zero in the case $n \geq 2$ (but $\sigma_z^0 \neq 0$), and are the following for n = 0 and n = 1 ($\sigma_z^0 = \sigma_\rho^0 + \sigma_\varphi^0$):

* In case n = 0 the torsion problem (with its characteristic roots $\mu_{0s}$) is completely divorced from that under consideration. Hence, here and henceforth, we understand $(-\gamma^{-2})\Delta_0(\gamma)$ to be a fourth order determinant which can be obtained by eliding the rows m = 3, 6 and the columns k = 3, 6 from the expanded matrix (6).

$$\frac{2(1-\nu)\Lambda_0(0)}{\Theta_0}\begin{pmatrix}\sigma_\rho^0\\ \sigma_\varphi^0\end{pmatrix}=j_0\begin{pmatrix}\left[\frac{a^2}{a^2-b^2}\left(1-\frac{b^2}{r^2}\right)\ln\frac{a}{b}-\ln\frac{r}{b}\right]\\ \left[\frac{a^2}{a^2-b^2}\left(1+\frac{b^2}{r^2}\right)\ln\frac{a}{b}-\ln\frac{r}{b}-1\right]\end{pmatrix}, \tag{14}$$

$$\frac{4(1-\nu)\Lambda_1(0)}{\Theta_1\cos\varphi}\begin{pmatrix}\sigma_\rho^0\\ \sigma_\varphi^0\end{pmatrix}=j_0\frac{a^2}{(a^2+b^2)}\frac{b}{r}\left(1-\frac{1}{\mathrm{Bi}_b}\right)\begin{pmatrix}\left(1-\frac{r^2}{a^2}\right)\left(1-\frac{b^2}{r^2}\right)\\ \left[4\frac{b^2}{a^2}-\left(3\frac{r^2}{a^2}-1\right)\left(1+\frac{b^2}{r^2}\right)\right]\end{pmatrix}, \tag{15}$$

$$\frac{4(1-\nu)\Lambda_1(0)}{\Theta_1\sin\varphi}\tau_{\rho\varphi}^0=j_0\frac{a^2}{(a^2+b^2)}\frac{b}{r}\left(1-\frac{1}{\mathrm{Bi}_b}\right)\left(1-\frac{r^2}{a^2}\right)\left(1-\frac{b^2}{r^2}\right), \tag{16}$$

$$\frac{2n\Lambda_n(0)}{\Theta_n\cos n\varphi}\sigma_z^0=-j_0\left[\left(\frac{r}{b}\right)^n\left(1+\frac{n}{\mathrm{Bi}_b}\right)-\left(\frac{b}{r}\right)^n\left(1-\frac{n}{\mathrm{Bi}_b}\right)\right]\quad(n\geqslant 2). \tag{17}$$

Here $j_0$ is the discontinuous factor

$$j_0=\frac{2}{\pi}\int_0^\infty\frac{\sin\gamma l\cos\gamma\xi}{\gamma}d\gamma=\begin{cases}1 & \xi<l\\ 1/2 & \xi=l\\ 0 & \xi>l,\end{cases} \tag{18}$$

and in conformity with (3), for $n = 0$ and $n \geq 1$ we have as $\gamma \to 0$:

$$\Lambda_0(0)=\left[\ln\frac{a}{b}+\frac{1}{\mathrm{Bi}_a}+\frac{1}{\mathrm{Bi}_b}\right], \tag{19}$$

$$\Lambda_n(0)=\frac{1}{2n}\left[\left(\frac{a}{b}\right)^n\left(1+\frac{n}{\mathrm{Bi}_a}\right)\left(1+\frac{n}{\mathrm{Bi}_b}\right)-\left(\frac{b}{a}\right)^n\left(1-\frac{n}{\mathrm{Bi}_a}\right)\left(1-\frac{n}{\mathrm{Bi}_b}\right)\right]. \tag{20}$$

The component $\Omega_N^T(\rho,\lambda,\xi)$, reflecting the local nature of the heating (1), in (13) is the following ($\xi \geq 0$) for $N = 1, 2, 3, 6$:

$$\Omega_N^T(\rho,\lambda,\xi)=\sum_{s=1}^\infty\{[\mathrm{res}\,F_N(\rho,\lambda,i\varkappa_s)][e^{-\varkappa_s(l+\xi)}+\varepsilon e^{-\varkappa_s(|l-\xi|)}]$$
$$+[\mathrm{res}\,F_N(\rho,\lambda,i\mu_s)][e^{-\mu_s(l+\xi)}+\varepsilon e^{-\mu_s(|l-\xi|)}]+2\,\mathrm{Re}[\mathrm{res}\,F_N(\rho,\lambda,\gamma_s)][e^{i\gamma_s(l+\xi)}+\varepsilon e^{i\gamma_s(|l-\xi|)}]\}, \tag{21}$$

where $\varepsilon$ is a factor taking on the following values

$$\varepsilon=1\ \text{for}\ \xi<l,\quad \varepsilon=0\ \text{for}\ \xi=l,\quad \varepsilon=-1\ \text{for}\ \xi>l. \tag{22}$$

The expressions for the residues in (21) are the following by virtue of (12), (7)-(9) and (3), for $N = 1$, say:

$$[\mathrm{res}\,F_1(\rho,\lambda,i\varkappa_s)]=\left\{\left[-\frac{1}{\gamma}[-]L(\gamma\rho,\beta)-\left[F_1^{(1)}\frac{\alpha}{\mathrm{Bi}_a}+F_1^{(2)}\right][-]L_\alpha'+\left[F_1^{(4)}\frac{\beta}{\mathrm{Bi}_b}-F_1^{(5)}\right]\frac{1}{\beta}\right]\frac{1}{\Lambda'(\gamma)}\right\}_{\gamma=i\varkappa_s}, \tag{23}$$

$$[\mathrm{res}\,F_1(\rho,\lambda,i\mu_s)]=\left[H^+(\lambda,i\mu_s)\frac{\sigma_\rho(\rho,\lambda,i\mu_s)}{\Delta'(\lambda,i\mu_s)}\right],$$
$$[\mathrm{res}\,F_1(\rho,\lambda,\gamma_s)]=\left[H^+(\lambda,\gamma_s)\frac{\sigma_\rho(\rho,\lambda,\gamma_s)}{\Delta'(\lambda,\gamma_s)}\right]. \tag{24}$$

On the basis of (12), (11), and (8), the notation $\sigma_\rho^{(1)}(\rho,\lambda,i\mu_s)=\sigma_\rho(\rho,\lambda,i\mu_s)$, $\sigma_\rho^{(1)}(\rho,\lambda,\gamma_s)=\sigma_\rho(\rho,\lambda,\gamma_s)$, . . . , as well as

$$H^+(\lambda,\gamma)=\left[([-]L)-f_{2:1}([-]L_\alpha')+f_{4:1}\left(\frac{1}{\mathrm{Bi}_b}\right)-f_{5:1}\left(\frac{1}{\beta}\right)\right]\frac{1}{\Lambda(\gamma)}, \tag{25}$$

which takes account of the linear dependence of the cofactors of the elements of rows in the system determinant $(-\gamma^{-2})\Delta(\lambda,\gamma)$ for values $\gamma = i\mu_s$ and $\gamma = \gamma_s = \eta_s + i\theta_s$ of its roots

$$\frac{A_{m1}(\lambda,\gamma)}{A_{11}(\lambda,\gamma)}=\ldots=\frac{A_{m6}(\lambda,\gamma)}{A_{16}(\lambda,\gamma)}=f_{m:1}(\lambda,\gamma)\equiv f_{m:1}, \tag{26}$$

is used in (24).

Here $\varepsilon \equiv -1$ in (21) for N = 4, 5, and quantities similar to (23) and (24) are subject to replacement by corresponding quantities multiplied by (–i) (see [1]).

It is seen from the structure of the solution (13) that members of the form (21) decrease as $l$ grows, and therefore, the distortion in the stress state caused by local heating (1) diminishes and vanishes as $l \to \infty$. The main members in (13) (see $\sigma_z^0$ and (14)-(16) for $\xi < l$), which describe the unperturbed stress state (when the medium temperature $T_a$ is constant along the axis over the whole length of the generatrix), are the solution of the appropriate plane problem of thermoelasticity [2] for a hollow cylinder not clamped at the endfaces* (the axial force and bending moment in any cross-section $\xi$ of the cylinder equals zero†).

Let us note that if the medium temperature at the inner surface of the cylinder is kept constant in both the axial and circumferential directions (i.e., $T_b$ = const ≠ 0) in (1)), then it is necessary to replace $T_0$ by the difference $T_0 - T_b$ in $\Theta_0$ of the expressions $\sigma_\rho^0$, $\sigma_\varphi^0$, $\sigma_z^0$ (14).

If the discontinuous temperature field in the cylinder originates because of uniform heating of a portion of its inner (r = b) side surface (i.e., $T_b$ is discontinuous in (1), $T_a = T(a, \varphi, z) = 0$), then to find the main terms of the solution of this problem it is sufficient to replace $a$ by b and b by $a$, $-\mathrm{Bi}_b$ by $\mathrm{Bi}_a$ in the right sides of (15)-(17), as well as to reverse the sign in front of the right sides in (14)-(17). The expressions for the residues [res $F_N$] in the additional terms in (21) are now to be subject to replacement by the expressions [res $F_N^-$], where (see (23)-(25)).

$$[\mathrm{res}\, F_1^-(\rho,\ \lambda,\ i\varkappa_s)] = [\mathrm{res}\, F_1(\rho,\ \lambda,\ i\varkappa_s)]\left[\beta\left\{\left[1 + \frac{\alpha}{\mathrm{Bi}_a}\frac{\partial}{\partial\alpha}\right]L_\beta'\right]_{\gamma=i\varkappa_s}, \tag{27}$$

$$[\mathrm{res}\, F_1^-(\rho,\ \lambda,\ i\mu_s)] = \left[H^-(\lambda,\ i\mu_s)\frac{\sigma_\rho(\rho,\ \lambda,\ i\mu_s)}{\Delta'(\lambda,\ i\mu_s)}\right],$$

$$[\mathrm{res}\, F_1^-(\rho,\ \lambda,\ \gamma_s)] = \left[H^-(\lambda,\ \gamma_s)\frac{\sigma_\rho(\rho,\ \lambda,\ \gamma_s)}{\Delta'(\lambda,\ \gamma_s)}\right], \tag{28}$$

$$H^-(\lambda,\ \gamma) = \left[\frac{1}{\mathrm{Bi}_a} + f_{2:1}\cdot\left(\frac{1}{\alpha}\right) + f_{4:1}\cdot([+]L) - f_{5:1}\cdot([+]L_\beta')\right]\frac{1}{\Lambda(\gamma)}. \tag{29}$$

Let us examine the axisymmetric heating case (n = 0), which is most important for applications, in greater detail.

The values of the roots of the known [3] characteristic equation (3), needed for numerical computations by means of (13) (for different values of the Biot criteria) are contained in tables [4-6].

The roots of the equation $-\gamma^{-2}\Delta_0(\lambda,\gamma)$ [7], which are in expanded form

$$-\gamma^{-2}\Delta_0(\gamma) = \frac{1}{\alpha\beta}\{[\alpha^2 + 2(1-\nu)][\beta^2 + 2(1-\nu)]L_{\alpha\beta}''^2$$

$$-\beta^2[\alpha^2 + 2(1-\nu)]L_\alpha'^2 - \alpha^2[\beta^2 + 2(1-\nu)]L_\beta'^2 + \alpha^2\beta^2L^2 + \alpha^2 + \beta^2 + 4(1-\nu)\} = 0, \tag{30}$$

are found by solving the appropriate elastic problem of a hollow cylinder and are presented in Table 1 ($\nu$ = 0.25).

The exact equation (30) can be replaced by an asymptotic equation for $|\beta| \geq 10$ on the basis of asymptotic representations of the Bessel functions:

$$-\gamma^{-2}\Delta_0(\gamma) \approx \frac{1}{\alpha\beta}\left\{(\alpha-\beta)^2 - \frac{1}{2}\,\mathrm{ch}\,2(\alpha-\beta) - \frac{\alpha-\beta}{\alpha\beta}\left[(1-\nu) - \frac{1}{8}\right]\mathrm{sh}\,2(\alpha-\beta)\right.$$

$$\left.-\frac{1}{\alpha\beta}\left[2(1-\nu)(\alpha-\beta)^2 - \frac{\alpha^2+\beta^2}{4}\right]\right\} = 0, \tag{31}$$

and its approximate solution is (for s ≥ 2):

$$\lambda\gamma_s \approx \frac{1}{2}\left(\ln t_s + i\frac{t_s}{2}\right) - \delta_s,\quad t_s = 2\pi s - \pi,$$

* Pure bending stresses, due to the moment, are excluded from consideration starting with (5) (for n = 1).
† The conditions that the axial force and bending moment be zero are satisfied not only for the main terms $\sigma_z^0$ but also for the whole expression (13) for $\sigma_z$.

TABLE 1. Values of the Roots of the Transcendental Equation (30) of the Axisymmetric Problem of an Elastic Hollow Cylinder

| $\frac{b}{a}$ | $\lambda$ | $\lambda\gamma_1=\lambda\eta_1+i\lambda\theta_1$ | $\lambda\gamma_2=\lambda\eta_2+i\lambda\theta_2$ | $\lambda\gamma_3=\lambda\eta_3+i\lambda\theta_3$ |
|---|---|---|---|---|
| 0,1 | 9/11 | 0,6353+*i*1,1787 | 1,0261+*i*2,7128 | 1,2800+*i*4,1154 |
| 0,2 | 2/3 | 0,6040+*i*0,9976 | 1,0798+*i*2,3793 | 1,3691+*i*3,9072 |
| 0,3 | 7/13 | 0,5681+*i*0,8420 | 1,1122+*i*2,2541 | 1,3809+*i*3,8276 |
| 0,4 | 3/7 | 0,5254+*i*0,7124 | 1,1213+*i*2,1873 | 1,3833+*i*3,7912 |
| 0,5 | 1/3 | 0,4771+*i*0,6014 | 1,1224+*i*2,1503 | 1,3812+*i*3,7692 |
| 0,6 | 1/4 | 0,4236+*i*0,5026 | 1,1234+*i*2,1276 | 1,3828+*i*3,7600 |
| 0,7 | 3/17 | 0,3638+*i*0,4100 | 1,1242+*i*2,1162 | 1,3838+*i*3,7542 |
| 0,8 | 1/9 | 0,2943+*i*0,3175 | 1,1248+*i*2,1100 | 1,3840+*i*3,7508 |
| 0,9 | 1/19 | 0,2065+*i*0,2139 | 1,1252+*i*2,1070 | 1,3842+*i*3,7492 |

TABLE 2. Change in the Quantity $\sigma_\varphi|_{z=0}/\sigma^0_\varphi$ on the Side Surfaces (r = $a$ and r = b) of Hollow Cylinders of Different Thickness as the Length (2c) of the Axisymmetrically Heated Portion of the Outer Surface (r = $a$) Increases (for $T_b$ = 0)

| b/a | | c/a | | | | |
|---|---|---|---|---|---|---|
| | | 0,1 | 0,3 | 0,5 | 1,0 | 1,5 |
| 0,2 | $r=a$ | 1,971 | 1,306 | 1,028 | 0,909 | 0,960 |
| | $r=b$ | 0,180 | 0,519 | 0,768 | 1,025 | 1,037 |
| 0,4 | $r=a$ | 1,524 | 1,064 | 0,901 | 0,897 | 0,968 |
| | $r=b$ | 0,211 | 0,573 | 0,831 | 1,271 | 1,040 |
| 0,6 | $r=a$ | 1,253 | 0,909 | 0,872 | 0,974 | 1,001 |
| | $r=b$ | 0,348 | 0,806 | 0,998 | 1,033 | 1,003 |
| 0,8 | $r=a$ | 1,163 | 0,851 | 0,826 | 0,972 | 1,006 |
| | $r=b$ | 0,331 | 0,817 | 0,969 | 1,046 | 1,001 |

$$t_s\delta_s=\frac{1}{t_s}\left[\ln^2 t_s+\frac{\lambda^2(7-8\nu)}{1-\lambda^2}(\ln t_s-1)+\frac{1}{2}\right]+i\left[\ln t_s-\frac{\lambda^2(7-8\nu)}{2(1-\lambda^2)}\right]. \tag{32}$$

It is not without interest to note that by keeping the main terms in the first two members of (31), we obtain the equations

$$\frac{4\lambda^2}{1-\lambda^2}\left[1-\frac{2\,\mathrm{ch}\,4\lambda\gamma}{(4\lambda\gamma)^2}\right]\approx\frac{4\lambda^2}{1-\lambda^2}\left(1+\frac{e^{2\lambda\gamma}}{4\lambda\gamma}\right)\left(1-\frac{e^{2\lambda\gamma}}{4\lambda\gamma}\right)=0, \tag{33}$$

which are identical to the well-known equations characteristic of problems on compression and bending of an elastic layer of thickness 2 λ [8]

$$1+\frac{\mathrm{sh}\,2\lambda\gamma}{2\lambda\gamma}=0,\quad 1-\frac{\mathrm{sh}\,2\lambda\gamma}{2\lambda\gamma}=0, \tag{34}$$

if $\exp(-4\lambda\gamma)$ in the last members are neglected in comparison with unity:

$$1+\frac{e^{2\lambda\gamma}}{4\lambda\gamma}=0,\quad 1-\frac{e^{2\lambda\gamma}}{4\lambda\gamma}=0. \tag{35}$$

There results from (33), (35), and (32) that the magnitude of the product $2\lambda\gamma_s$ is almost independent of λ and for s = 2, 4, . . . agrees with the roots of the equation for a compressed, and for s = 3, 5, . . . of a cambered layer, where $2\lambda\gamma_s$ is practically a constant (for fixed s) for $b/a \geq 0.4$ and $S \gtrsim 2$ (see Table 1).

To illustrate the order of the computation, let us find values of the quantities $\sigma_\varphi$ and $\sigma_z$ on the side surfaces r = $a$ and r = b in the z = 0 section of a cylinder having zero temperature on the inner surface and being heated on a portion of the outer surface of length 2c = 0.5$a$, say, if b = 0.4$a$ and $\mathrm{Bi}_a = \mathrm{Bi}_b = \infty$.

Using the data of Table 1 and [6],* as well as values of functions of the type (23), (24) which we have tabulated, we obtain by means of (13), (14), (18), (19), (21), (22) for $\rho = 1 \pm \lambda$ (to verify the rapid convergence of the series, the summed terms s = 1, 2, 3 are shown)

* The asymptotic representation of the roots of (3), which here degenerates into $\Lambda_0(\lambda,\gamma) = L_0 = 0$, is the following: $\lambda\varkappa_s\approx\frac{\pi s}{2}\left[1-\frac{\lambda^2}{2(1-\lambda^2)}\frac{1}{\pi^2 s^2}\right]$.

$$\frac{(\sigma_\varphi)_a}{(\sigma_\varphi^0)_a} = 1 - 4.223\{[-0.014 + 2(-0.010)] + [-0.001 + 2\cdot 0.001] + 0\} = 1.139, \tag{36}$$

$$\frac{(\sigma_\varphi)_b}{(\sigma_\varphi^0)_b} = 1 + 2.326\{[-0.058 + 2(-0.138)] + [-0.004 + 2\cdot 0.002] + [0.001 - 0]\} = 0.495, \tag{37}$$

$$(\sigma_z)_a = -0.005\,(\sigma_z^0)_a, \qquad (\sigma_z)_b = 0.214\,(\sigma_z^0)_b, \tag{38}$$

where

$$(\sigma_\varphi^0)_a = (\sigma_z^0)_a = \frac{\Theta_0}{2(1-\nu)}\left[\frac{2b^2}{a^2-b^2} - \frac{1}{\ln\frac{a}{b}}\right];$$
$$(\sigma_\varphi^0)_b = (\sigma_z^0)_b = \frac{\Theta_0}{2(1-\nu)}\left[\frac{2a^2}{a^2-b^2} - \frac{1}{\ln\frac{a}{b}}\right]. \tag{39}$$

Starting from the Hooke's law relationships with temperature terms

$$Eu = r\{[\sigma_\varphi - \nu(\sigma_z + \sigma_\rho)] + \alpha_T ET\},$$

we determine the radial displacements of the side surfaces (on which $\sigma_\rho = 0$) in the same z = 0 section, in passing:

$$(u)_a = 0.715\,(u^0)_a, \quad (u)_b = 0.580\,(u^0)_b,$$

wherein

$$\frac{Eu^0}{\Theta_0} = j_0\frac{r}{2}\left\{\frac{2a^2}{a^2-b^2} - \frac{1}{\ln\frac{a}{b}} - \frac{1+\nu}{1-\nu}\left[\frac{a^2}{a^2-b^2}\left(1-\frac{b^2}{r^2}\right) - \frac{\ln\frac{r}{b}}{\ln\frac{a}{b}}\right]\right\}.$$

Presented in Table 2 are results of calculating the values of the ratios of the circumferential stresses (see (36)-(37)), found for cylinders of different thickness for a different relative length of the heated portion of the outer surface ($\lim_{c\to\infty} \sigma_\varphi = \sigma_\varphi^0$). These results afford the possibility of quantitative estimation of the distortion induced by local heating in the values of the stresses $\sigma_\varphi^0$ (39) which are constant over the whole generator length under uniform axisymmetric heating of the outer side surface of the cylinder.

It is easy to go from the solution (13) over to the solution for the case of concentrated heating (in the section $\xi = 0$) on a circle of radius $\rho = 1 + \lambda$ :

$$\lim 2lT_n = T_n^* R^{-1} \qquad (2l \to 0, \quad T_n \to \infty), \tag{40}$$

where $T_n^*$ is the temperature per unit length of the outer circumference of the cylinder.

The passage to the limit in (21) (for $\xi > l$) results in the following expressions for the functions $\omega_N^T(\rho, \lambda, \xi)$ for N = 1, 2, 3, 6:

$$\omega_N^T(\rho, \lambda, \xi) = \sum_{s=1}^{\infty}\{[-\varkappa_s \operatorname{res} F_N(\rho, \lambda, i\varkappa_s)]\exp(-\varkappa_s\xi)$$
$$+ [-\mu_s \operatorname{res} F_N(\rho, \lambda, i\mu_s)]\exp(-\mu_s\xi) + 2\operatorname{Re}[i\gamma_s \operatorname{res} F_N(\rho, \lambda, \gamma_s)]\exp(i\gamma_s\xi)\}, \tag{41}$$

and in an analogous form for N = 4, 5.

The expressions for the stresses for the case of concentrated (40) heating (1) can now be represented thus for example ($\Theta^* \equiv \Theta_n^* = [\alpha_T E T_n^*]$):

$$\frac{R\sigma_\rho}{\Theta^*} = \omega_1^T \cos n\varphi, \quad \frac{R\sigma_z}{\Theta^*} = \omega_3^T \cos n\varphi, \quad \frac{R\tau_{\varphi z}}{\Theta^*} = \omega_5^T \sin n\varphi. \tag{42}$$

The results obtained above can easily be extended, on the basis of the superposition principle, to the case of heating a portion of the outer side surface of a cylinder according to any law $T_a = T(a, \varphi, z)$.

Taking account of all that has gone before (see (27)-(29)), the passage to the solution for the case of arbitrary heating on the inner surface $T_b = T(b, \varphi, z)$ is also quite simple.

## NOTATION

| | |
|---|---|
| $T = T(r, \varphi, z)$ | is the temperature at a point of an elastic cylinder; |
| $a$ | is the outer radius; |
| b | is the inner radius; |
| $R = (a + b)/2$ | is the mean radius of the cylinder; |
| $\lambda = (a - b)/2R$ | is half the relative thickness of the cylinder wall; |
| $l = c/R$ | is half the relative length of the heated section; |
| $\rho = r/R\ (1-\lambda \le \rho \le 1+\lambda)$; | |
| $\xi = z/R$ | are the dimensionless coordinates; |
| $\alpha = \gamma a/R = \gamma(1+\lambda)$; | |
| $\beta = \gamma b/R = \gamma(1-\lambda)$ | are the values of the quantity $\gamma\rho$ for $\rho = 1 \pm \lambda$; |
| $h_a$, $h_b$ | are the relative heat transfer coefficients; |
| $Bi_a = ah_a$; | |
| $Bi_b = bh_b$ | are the Biot criteria; |
| $\alpha_T$ | is the coefficient of linear thermal expansion; |
| E | is the elastic modulus; |
| $\nu$ | is the Poisson ratio; |
| $[+] \equiv [1 + (\alpha/Bi_a)(\partial/\partial\alpha)]$, | |
| $[-] \equiv [1 - (\beta/Bi_b)(\partial/\partial\beta)]$ | are the operators; |
| $L \equiv L(x, y) \equiv L_n(x, y)$ | |
| $= I_n(x)K_n(y) - I_n(y)K_n(x)$ | are the combinations of modified Bessel functions of the first $I_n(x) \equiv I(x)$ and third $K_n(x) \equiv K(x)$ kinds of order n, where $\{L'_x(x, y)\}_{y=x} = 1/x$ (the prime denotes differentiation with respect to the argument); |
| $V_I(x) = (1-2\nu)I(x) + xI'(x)$; | |
| $W_I(x) = V'_I(x) + I'(x) - n^2 I(x)/2x$. | |

## LITERATURE CITED


1. P. Z. Livshits, Inzh.-Fizich. Zh., 16, No. 4 (1969).
2. L. Bally and G. Weiner, Theory of Temperature Stresses [Russian translation], Mir (1964).
3. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
4. B. R. Belostotskii, Inzh.-Fizich. Zh., 13, No. 3, 326-329 (1967).
5. R. G. Geints, Inzh.-Fizich. Zh., 5, No. 10, 77-81 (1962).
6. E. Jahnke, F. Emde, and F. Loesch, Special Functions [Russian translation], Nauka (1968).
7. V. K. Prokopov, Prikl. Matem. i Mekh., 13, No. 2, 135-144 (1949).
8. A. I. Lur'e, Three-Dimensional Problems of Elasticity Theory [in Russian], Moscow, GITTL (1955).